MATH 1A - A DETAILED EXPLANATION OF THE PROOF OF THE FUNDAMENTAL THEOREM OF CALCULUS

PEYAM RYAN TABRIZIAN

1. The Fundamental Theorem of Calculus

Just to remind you, this is the statement of the Fundamental Theorem of Calculus. Remember that there are two versions. We will prove both versions, but Part II is much easier to prove than Part I.

Theorem 1 (Fundamental Theorem of Calculus - Part I)**.** *If f is continuous on $[a,b]$, then the function g defined by:*

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is continuous on $[a,b]$, differentiable on (a,b) and $g'(x) = f(x)$ for all $x \in (a,b)$.

Theorem 2 (Fundamental Theorem of Calculus - Part II)**.** *If f is continuous on $[a,b]$, then:*

$$\int_a^b f(t)dt = F(b) - F(a)$$

where F is any antiderivative of f

2. Proof of FTC - Part I

This is probably one of the longest and hardest proofs you'll ever see in this class, and probably in your whole mathematics career. If you understand this, then you're truly the Jedi-Master of Calculus! What's really beautiful about this proof, though, is that it combines nearly all the concepts we've discussed this semester, namely limits, ϵ, δ, differentiation, and integration. So be prepared for a helluva ride :)

Warning: This version is very 'scrambled' up, and you should use this handout only to **understand** how we get the proof. If you're curious in how to actually write up the proof on the exam, refer to the undetailed handout "Proof of the FTC".

Now let's start! The really important part of the theorem is proving that $g'(x) = f(x)$ for all $x \in (a,b)$. So as usual, let x be in (a,b). Usually, in those type of hard proofs, we first write what we want to compute. Ultimately, we want to show that $\boxed{g'(x) = f(x)}$. So in this case, we first want to compute $g'(x)$, which is defined as:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Date: Wednesday, November 17th, 2010.

Our strategy here is: Compute $\frac{g(x+h)-g(x)}{h}$, and **then** take the limit! We did this a bunch of times when we computed limits!

So, by definition of g, we get:

$$\frac{g(x+h)-g(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

Here, we use the fact that $\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt$, so subtracting $\int_a^x f(t)dt$ from both sides, we get: $\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$

So, what we really want to show is:

$$\lim_{h\to 0} \frac{\int_x^{x+h} f(t)dt}{h} = f(x)$$

Now here's a trick that saves us half of the work: Assume $h > 0$, and to show that $\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)dt}{h} = f(x)$. I know, it looks like cheating, but your proof is actually valid, if you say the following magic phrase: **the case $h < 0$ is similar**. The point is that to show that a limit exists, it suffices to show that the left-hand-limit ($h \to 0^-$) and the right-hand-limit ($h \to 0^+$) exist and are equal. Here we are only showing that the left-hand-limit exists and equals $f(x)$, **but** the proof that the right-hand-limit exists and equals $f(x)$ is **ALMOST EXACTLY THE SAME**. So instead of writing down the whole proof again, you just say the magic phrase, and you are safe!

Alright, great, so now how do we compute this mysterious limit? None of the techniques we know work!!! We are stuck, and yes, unfortunately there is only one way out, and it is the thing that has haunted your dreams until the first midterm: $\epsilon - \delta$ (dramatic music playing in the background, **TUM TUM TUUUUUUUUUUUM!!!**)! Trust me, I've been trying to find ways to avoid this, but here, we simply have to face our enemy with our weapons!

So, as usual with $\epsilon - \delta$-arguments, we let $\epsilon > 0$. The point is to find $\delta > 0$ such that when $0 < h < \delta$ (remember $h > 0$), then:

$$\left| \frac{\int_x^{x+h} f(t)dt}{h} - f(x) \right| < \epsilon$$

Now this looks terrible!!! How the hell are you supposed to show this? And what should δ be? (remember that you have to **find** δ!) But wait!!! There is one **important** assumption we haven't used yet: f is continuous!!! It seems like a small assumption, but it is actually crucial to this proof! **The FTC doesn't work when f is not continuous!**

Now, f is continuous on $[a, b]$, f is also continuous at x, so there exists $\delta > 0$ such that, when $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$.

Now here comes an important step, one which I call 'sneaking behind one's back'-method. I will basically start with the fact that f is continuous at x, and then get more and more complicated statements, and finally **BANG!**, we get the result out of nowhere! This is a very indirect proof, and you might not be used to such a proof-strategy!

So let's start with this method:

As I said, we'd like to use the fact that f is continuous, and for this, we need to somehow get $|t - x| < \delta$. Notice that if t is in $[x, x+h]$, we have $x \leq t \leq x+h$, so $0 \leq t - x \leq h < \delta$, and so in particular $|t - x| = t - x < \delta$, which is exactly what we want, because we get $|f(t) - f(x)| < \epsilon$.

But remember that in general, $|a| < \epsilon$ is the same as $-\epsilon < a < \epsilon$. So with $a = f(t) - f(x)$, this implies that $-\epsilon < f(t) - f(x) < \epsilon$, so $f(x) - \epsilon < f(t) < f(x) + \epsilon$.

Great, so now all need to do is to integrate what we just found over the interval $[x, x+h]$ and use the comparison inequalities, and we get:

$$f(x) - \epsilon < f(t) < f(x) + \epsilon$$
$$\int_x^{x+h} (f(x) - \epsilon)\, dt < \int_x^{x+h} f(t)dt < \int_x^{x+h} (f(x) + \epsilon)\, dt$$

However, the point is that $f(x) - \epsilon$ and $f(x) + \epsilon$ **DO NOT** depend on t, so they are **CONSTANTS** with respect to t, and we can pull them out of the integral! And so we get:

$$(f(x) - \epsilon) \int_x^{x+h} dt < \int_x^{x+h} f(t)dt < (f(x) + \epsilon) \int_x^{x+h} dt$$
$$(f(x) - \epsilon)(x + h - x) < \int_x^{x+h} f(t)dt < (f(x) + \epsilon)(x + h - x)$$

Here, we used the fact that $\int_x^{x+h} dt = \int_x^{x+h} 1dt = x+h-x$, and because $x+h-x = h$, we get

$$(f(x) - \epsilon)\, h < \int_x^{x+h} f(t)dt < (f(x) + \epsilon)\, h$$

And dividing everything by h, keeping in mind that $h > 0$, we get:

$$(f(x) - \epsilon) < \frac{\int_x^{x+h} f(t)dt}{h} < (f(x) + \epsilon)$$

And subtracting $f(x)$ from all sides, we get:

$$-\epsilon < \frac{\int_x^{x+h} f(t)dt}{h} - f(x) < \epsilon$$

Finally, remember that $|a| < \epsilon$ is the same as $-\epsilon < a < \epsilon$, we get that:

$$\left| \frac{\int_x^{x+h} f(t)dt}{h} - f(x) \right| < \epsilon$$

Oh my! Just look at what we've done!!! We started out with a random fact (that f is continuous), and we applied what we know to get our desired result!!!

And so we've shown that:

$$\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)dt}{h} = f(x)$$

Which is the same as:

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = f(x)$$

Similarly (the magic phrase), one can show that:

$$\lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} = f(x)$$

And hence, we get:

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$

But: $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$ by definition of $g'(x)$
And so, we finally have:

$$g'(x) = f(x)$$

And Abra-Kadabra-Alakazam, we're done! :D

So just to re-cap, here is what we've done:

(1) We noticed that $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$
(2) We computed $\frac{g(x+h)-g(x)}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$
(3) We said that it is enough to consider the case $h > 0$, and all we need to prove is that $\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)dt}{h} = f(x)$
(4) Then, we started with the fact that f is continuous, and using this and integrating and some algebra, we magically got what we wanted to show
(5) And this gave us our result, because the case $h \to 0^-$ is similar and because $\frac{\int_x^{x+h} f(t)dt}{h} = \frac{g(x+h)-g(x)}{h}$

3. Proof of FTC - Part II

This is much easier than Part I!

Let F be an antiderivative of f, as in the statement of the theorem.

The idea of this proof is to define two new functions which basically solve our problem (it's like hiring two maids to clean up your room), and to apply FTC Part I.

So define a new function g as follows:

$$g(x) = \int_a^x f(t)dt$$

This looks just like the function g defined above! In fact, by FTC Part I, g is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$ for every x in (a, b).

Now define **another** new function H as follows:

$$h(x) = g(x) - F(x)$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) because g and F are. We want to show that $h'(x) = 0$ (you may wonder why, but stick around, and you'll see why!).

Moreover, if $x \in (a, b)$, $h'(x) = g'(x) - F'(x)$, but what do those terms mean? We have $g'(x) = f(x)$ by FTC Part I, and $F'(x) = f(x)$ because F is an antiderivative of f. And so $h'(x) = f(x) - f(x) = 0$ **for every** $x \in (a, b)$, and so, h is constant on (a, b).

However, we'd like h to be constant on $[a, b]$ (with a and b included). **This is why you need the fact that h is continuous at a and b!** So, because h is also continuous at a and b, we get that h is constant on $[a, b]$.

And so, because h is constant on $[a, b]$, we get, in particular that $h(a) = h(b)$. But what does that mean? Here is what it means, and it ultimately solves our problem!

$$h(b) = h(a)$$
$$g(b) - F(b) = g(a) - F(a)$$

Where we have used the definition of h. Now, adding $F(b)$ from both sides, we get:

$$g(b) = g(a) + (F(b) - F(a))$$

And so, by definition of g, we get:

$$\int_a^b f(t)dt = \int_a^a f(t)dt + (F(b) - F(a))$$

But $\int_a^a f(t)dt = 0$ (the endpoints are the same), and so:

$$\int_a^b f(t)dt = F(b) - F(a)$$

TA-DAAAAA!!! And we're done!

So again, just by defining g and h and using FTC Part I, we got our desired result!